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CLASSIFYING 3-TRIP LORENZ KNOTS

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The topological types of closed periodic solutions of the Lorenz equations are in one-to-one correspondence with aperiodic positive words on two generators. The number of syllables in such a word is called the trip number of the corresponding knot. Classifications for knots with trip numbers 1 and 2 are known. This paper gives a complete classification for 3-trip Lorenz knots.

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Lorenz knots   strange attractors   periodic orbits

The topological properties of solutions to differential equations have recently come under study. In particular, Birman and Williams [1] have examined the Lorenz equations

$$\dot{x} = -10x + 10y,$$

$$\dot{y} = 24x - y - xy,$$

$$\dot{z} = -\tfrac{8}{3}z + xy.$$

For general background on this system the reader is referred to [3, 5, 6]. The attractor of the flow which arises from this system can be modelled by T, herein called a *template*, which topologically looks like Fig. 1. Note that T is a surface (with boundary) except along a branch set denoted $\overline{AB}$. The arrows indicate a semi-flow on T which is the projection of the actual flow. Any periodic orbit of the flow will project onto a simple closed curve on T which follows the semi-flow.

As a result of the above we make the following definition.

Definition. A *Lorenz knot* is a knot which has the knot type of a simple closed curve in T which respects the direction of the semi-flow.

By associating the symbol x with loops about the left-hand hole in T and y with loops about the right-hand hole, we can state one of the basic results of [1]:

Theorem. (Birman and Williams). *Lorenz knots are in* 1-1 *correspondence with the cyclic permutation classes of finite aperiodic words in x and y having positive exponents.*

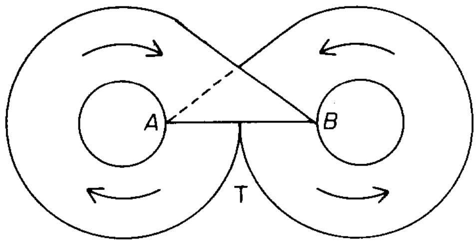

Fig. 1.

As an example of this theorem consider the word $w = xy^2x^2yx^2y^2$. The corresponding knot is shown in Fig. 2.

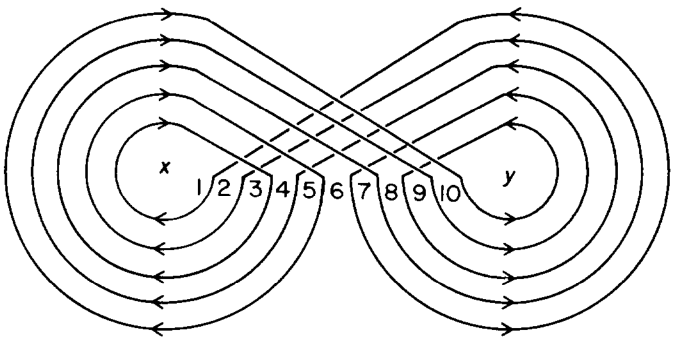

Fig. 2.

Reading the word from a given knot is easily done. The construction in the opposite direction requires some work. To do this the 10 cyclic permutations of w are formed and lexicographically numbered as shown below:

$xy^2x^2yx^2y^2$	4	$yx^2y^2xy^2x^2$	7
$y^2x^2yx^2y^2x$	9	$x^2y^2xy^2x^2y$	2
$yx^2yx^2y^2xy$	6	$xy^2xy^2x^2yx$	5
$x^2yx^2y^2xy^2$	1	$y^2xy^2x^2yx^2$	10
$xyx^2y^2xy^2x$	3	$yxy^2x^2yx^2y$	8

We mark off 10 points on the branch set as shown in Fig. 2. These are then connected in the order given by the lexicographical ordering. If the word begins with x (resp. y) we go around the left (resp. right) hole. (i.e., 4 is connected to 9 by an x loop, then 9 is connected to 6 by a y loop etc.).

It is easy to see that different words can represent the same knot type. For example both x^2y and xy^3 represent the trivial knot. We are thus led to ask: What knot types can arise as Lorentz knots? That is, can we write a complete list of distinct Lorenz knots? A useful tool in organizing such a list is given by the following definition:

Definition. The *trip number* of a Lorenz knot is the minimum number of syllables in any of its associated words. Geometrically this is just the number of times that the knot crosses from left to right in T.

Birman and Williams show the following where K is a Lorenz knot:

Theorem. *K has trip number $1 \Leftrightarrow K$ is a trivial knot.*

Theorem. *K has trip number $2 \Leftrightarrow K$ is a torus knot of type $(2, 2n+1)$.*

The purpose of this paper is to give a classification of Lorenz knots with trip number 3.

Notation and definitions. (1) Every 3-trip word can be written in the form $w = x^{a_1}y^{b_1}x^{a_2}y^{b_2}x^{a_3}y^{b_3}$. We will henceforth denote this word by the symbol $(a_1, b_1, a_2, b_2, a_3, b_3)$.

(2) We will say $w \sim w'$ if the associated knots are of the same type.

The first theorem cited includes the following in its statement:

Lemma 1. $(a_1, b_1, a_2, b_2, a_3, b_3) \sim (b_1, a_2, b_2, a_3, b_3, a_1)$.

In addition, Williams and Franks [2] have shown:

Lemma 2. $(a_1, b_1, a_2, b_2, a_3, b_3) \sim (b_3, a_3, b_2, a_2, b_1, a_1)$.

Remark. The proof of Lemma 2 involves looking closely at the way in which the knots are related to the system of equations and is beyond the scope of this paper. We note that the results which follow are independent of Lemma 2 but the proofs are somewhat longer without it.

Notation and definitions. (1) $\text{length}(w) = \sum a_i + \sum b_i$.

(2) A word w is *reducible* if $w \sim w'$ and $\text{length}(w') < \text{length}(w)$.

(3) Let $A = \{a_1, a_2, a_3\}$ and $B = \{b_1, b_2, b_3\}$, then denote

$$a_m = \max\{A\}, \quad a_n = \min\{A\}, \quad b_m = \max\{B\} \text{ and } b_n = \min\{B\}.$$

(4) Henceforth $a' > a$ where $a, a' \in A$ and $b' > b$ where $b, b' \in B$.

Lemma 3. *A word w which meets any of the following conditions can be reduced by replacing a_m by $a_m - 1$:*

(a) *a_m is unique in A and $a_m - 1 > a_i$ $\forall i \neq m$.*

(b) *a_m and b_n are unique in A and B respectively and a_m immediately precedes b_n in w.*

(c) $w = (a'+1, b, a', b, a, b)$, $(a+1, b, a, b, a, b+1)$, $(a'+1, b, a, b, a', b+1)$ *or* $(a'+1, b', a, b, a', b'+1)$.

(d) *w is the reverse or a cyclic permutation of any of the words in parts* (a) *to* (c).

Proof. (a) to (c) In the list of cyclic permutations of w, the word $w_1 = x^{a_m} \cdots$ immediately precedes the word $w_2 = x^{a_m - 1} \cdots x$. In each of the cases (a) to (c) w_1 and w_2 are the first two words in the lexicographical ordering. Hence the knot looks locally like Fig. 3(a). This is clearly isotopic to Fig. 3(b) which is represented by w' where a_m has been replaced by $a_m - 1$.

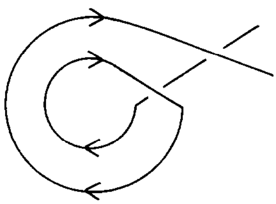

Fig. 3a

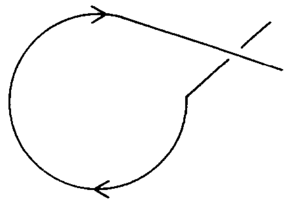

Fig. 3b

(d) Simply apply Lemmas 1 and/or 2 to get w into the form of one of cases (a) to (c). □

In practice, Lemmas 1 to 3 are often used together in a sequence of reductions as illustrated in the following example.

Example.

$(1, 3, 4, 1, 1, 2) \sim$
$\sim (1, 3, 2, 1, 1, 2)$ (by two applications of Lemma 3(a))
$\sim (1, 3, 1, 1, 1, 2)$ (by Lemma 3(b))
$\sim (2, 1, 1, 1, 3, 1)$ (by Lemma 2)
$\sim (3, 1, 2, 1, 1, 1)$ (by four applications of Lemma 1)
$\sim (2, 1, 2, 1, 1, 1)$ (by Lemma 3(c) (first type)).

Further reductions can be made by representing a knot as a closed braid as described in [1]:

Theorem (Birman and Williams). *Every 3-trip Lorenz knot can be represented as a closed positive 3-string braid.*

Recall that the braid group on three strings has the following presentation:

$$\langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle.$$

In the proof of the above theorem the braid of a Lorenz knot is shown to be of the form

$$(\sigma_1\sigma_2)^3(\sigma_1)^{n_1}(\sigma_1\sigma_2)^{n_2}(\sigma_2)^{m_2}(\sigma_2\sigma_1)^{m_1},$$

where the n_i and m_i can be computed directly from the word as follows: We return to the list of cyclic permutations referred to earlier. If two adjacent words begin with x (resp y) we define their *x-skip* (*y-skip*) as the unsigned difference between their positions in the lexicographical ordering. The n_i and m_i are then defined by

n_i = the number of *x-skips* of length $i+1$,
m_i = the number of *y-skips* of length $i+1$.

In the example given earlier, the word $xy^2x^2yx^2y^2$ is represented by the braid

$$(\sigma_1\sigma_2)^3(\sigma_1)^1(\sigma_1\sigma_2)^1(\sigma_2)^1(\sigma_2\sigma_1)^1.$$

By applying this construction we can show that different words represent the same knot by computing their braid representations and showing that they are braid equivalent.

Lemma 4. *If* $a' > a+1$,

$$(a', b', a', b', a, b) \sim (a'-1, b', a', b', a, b).$$

Proof. The braid representing the first word is

$$(\sigma_1\sigma_2)^3(\sigma_1)^{2(a'-a)-2}(\sigma_1\sigma_2)^{3a-2}(\sigma_1)^{2(b'-b)}(\sigma_2\sigma_1)^{3b-3}.$$

This is braid equivalent to

$$\sigma_1(\sigma_2\sigma_1)^3(\sigma_1)^{2(a'-a)-3}(\alpha_1\sigma_2)^{3a-2}(\sigma_2)^{2(b'-b)}(\sigma_1\sigma_2)^{3b-3}$$

which is again equivalent to

$$(\sigma_1\sigma_2)^3(\sigma_1)^{2(a'-a)-3}(\sigma_1\sigma_2)^{3a-2}(\sigma_2)^{2(b'-b)-1}(\sigma_2\sigma_1)^{3b-2}.$$

This braid represents the second word in the lemma. □

The same technique is used to prove all of the remaining lemmas. Since the proofs are virtually identical we will not write out the details.

Lemma 5.

$$(a+1, b, a, b+1, a, b) \sim (a, b, a, b+1, a, b).$$

Lemma 6. *If* $b' > b+1$,

$$(a, b', a, b', a, b) \sim (a+1, b'-1, a+1, b'-1, a, b),$$

where the latter of these two words is reducible by Lemma 4.

We now pause to summarize our progress after applying all of the above lemmas to the set of all 3-trip words.

Proposition 1. *Every* 3*-trip word is equivalent to one of the following*:
Type Ia. $(a, b, a, b, a, b+1)$,
Type Ib. $(a, b, a, b+1, a, b+1)$,
Type IIa. $(a, b, a, b', a+1, b')$,
Type IIb. $(a, b, a', b'+1, a', b')$,
Type III. (a', b, a', b', a, b').

While none of the above is apparently reducible, there are several equivalences which do not reduce the length of the word. These are again proved by comparing the braid representatives of the corresponding words.

A Type IIb word can be changed into a Type IIa word of the same length by:

Lemma 7. *If* $a' > a+1$ *then*

$$(a, b, a', b'+1, a', b') \sim (a, b, a'-1, b'+2, a'-1, b'+1)$$

and if $a' = 1+1$,

$$(a, b, a+1, b'+1, a+1, b') \sim (a, b, a, b'+1, a+1, b'+1).$$

All words can be modified within their own type and without changing their lengths by subtracting 1 from each of the x exponents and adding 1 to each of the y exponents. We state this as:

Lemma 8. *If* $a > 1$ *then*
Type Ia. $(a, b, a, b, a, b+1) \sim (a-1, b+1, a-1, b+1, a-1, b+2)$,
Type Ib. $(a, b, a, b+1, a, b+1) \sim (a-1, b+1, a-1, b+2, a-1, b+2)$,
Type II. $(a, b, a, b', a+1, b') \sim (a-1, b+1, a-1, b'+1, a, b'+1)$,
Type III. $(a', b, a', b', a, b') \sim (a'-1, b+1, a'-1, b'+1, a-1, b'+1)$.

By repeated applications of this lemma we may assume that $a = 1$ in all types.

Finally, Type III words can be modified by:

Lemma 9.

$$(a', b, a', b', 1, b') \sim (b'-b+1, b, b'-b+1, a'+b-1, 1, a'+b-1).$$

Note that if $a' \geqslant b'$ then $a'+b-1 \geqslant b'-b+1$. Thus we may assume $a' \leqslant b'$.

We are now ready to state our principal result.

Theorem 1. *Every* 3*-trip Lorenz knot is equivalent to one and only one of the following*:
Type Ia. $(1, b, 1, b, 1, b+1)$.
Type Ib. $(1, b, 1, b+1, 1, b+1)$,

Type II. $(1, b, 1, b', 2, b')$,
Type III. $(a', b, a', b', 1, b')$ *where* $a' \leq b'$ *and* $b < b'$.

Remarks. (a) A word of Type Ia represents a torus knot of type $(3, 3b+1)$.
(1) A word of Type Ib represents a torus knot of type $(3, 3b+2)$.
(3) If length$(w) = L \equiv 0 \pmod 3$ there is no Type I word of length L.
(4) When $b = 1$ a Type II word represents a pretzel knot of type $(2, -3, -2b'-1)$
(5) When $b = 1$ a Type III word represents a pretzel knot of type $(2, -2a'-1, -2b'-1)$.

Proof. To show that any word w is equivalent to one of the above, we first reduce w by Lemmas 1 to 6 so that w is in one of the types in Proposition 1. We then modify w so that $a = 1$ and if necessary apply Lemma 9 so that $a' \leq b'$.

To show that w is equivalent to *only one* of the types in the theorem we will show that each of the types is distinct from the others and that within types the numbers a', b, and b' distingusih the knot type. To do this we will compute the Alexander polynomial of each type using a method described by Marasugi [4, Section 3]. We first put the braid representative of each type in a standard form where $\Delta = (\sigma_1\sigma_2\sigma_1)$.
Type Ia. $\Delta^{2b}\sigma_1\sigma_2$.
Type Ib. $\Delta^{2b}(\sigma_1\sigma_2)^2$,
Type II. $\Delta^{2(b+1)})\sigma_1^{-1}\sigma_2^{2(b'-b)-1}$,
Type III. $\Delta^{2b}\sigma_1^{2(a-1)+1}\sigma_2^{2(b'-b)+1}$.

We now denote the Alexander polynomial of the knot associated to w as $\Delta_w(t)$. Using [4, equation (3.7)] we have:

Type Ia.

$$\Delta_w(t) = \frac{t-1}{t^3-1}\left(\frac{1-t^{3(3b+2)}}{1-t^{3b+1}}\right),$$

Type Ib.

$$\Delta_w(t) = \frac{t-1}{t^3-1}\left(\frac{1-t^{3(3b+2)}}{1-t^{3b+2}}\right),$$

Type II.

$$\Delta_w(t) = \frac{t-1}{t^3-1}(t^{2(b'-2b+2)} + t^{2(b'-2b+2)-(3b+4)} - t^{2(b'-2b+2)-(3b+4)-1} + \cdots + t^{3b+4} + 1),$$

Type III.

$$\Delta_w(t) = \frac{t-1}{t^3-1}(t^{2(a+b'+2b)} - t^{3b+1}(-t^{2(b'-b)} - t^{2(a-1)}) + (t^{2(a-1)} - t^{2(a-1)-1} + \cdots + 1)(t^{2(b'-b)} - t^{2(b'-b)-1} + \cdots + 1)) + 1).$$

Murasugi shows that Type I $\cap$ Type II = Type I $\cap$ Type III = $\emptyset$. By expanding Type II and Type III polynomials we see that the polynomials in Type II have only ± 1 coefficients while all of those in Type III have at least one non-unitary coefficient, thus Type II $\cap$ Type III = $\emptyset$ as well. Within types we know that the knots of Type I are all torus knots and are thus distinguished from each other by the exponent b. Within Type II we see from the polynomials that if $w \sim \bar{w}$ then

$$2(b' - 2b + 2) = 2(\bar{b}' - 2\bar{b} + 2) \quad \text{and} \quad 3b + 4 = 3\bar{b} + 4,$$

which imply

$$b = \bar{b} \quad \text{and} \quad b' = \bar{b}'.$$

Similarly, albeit with a little more work, in Type III if $w \sim \bar{w}$

$$2(a + b' + 2b) = 2(\bar{a} + \bar{b}' + 2\bar{b}), \qquad 3b + 1 = 3\bar{b} + 1$$

and

$$2(a - 1) = 2(\bar{a} - 1) \quad \text{and} \quad 2(b' - b) = 2(\bar{b}' - \bar{b}).$$

These equations imply

$$a = \bar{a}, \qquad b = \bar{b} \quad \text{and} \quad b' = \bar{b}'.$$

Note that we might conclude from the polynomials that

$$2(a - 1) = 2(\bar{b}' - \bar{b}) \quad \text{and} \quad 2(b' - b) = 2(\bar{a} - 1).$$

In this case however, w and $\bar{w}$ are of the two types which are shown equivalent by Lemma 9 and we have already chosen the one in which $a' \leqslant b'$ and we are back in the former case. □

Many of the equivalences used to prove the thorem were discovered using a computer program which computes the Alexander polynomial of a Lorenz knot from its defining word. The program works for words of any trip number and for knots on a generalized template with twists in the left and right bands. It is hoped that some kind of general pattern similar to the one given above can be discovered for these *template knots*.

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